

Harmonic approximation and improvement of flatness in a singular perturbation problem

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Abstract

We study the De Giorgi type conjecture, that is, one dimensional symmetry problem for entire solutions of an two components elliptic system in \mathbb{R}^n , for all $n \geq 2$. We prove that, if a solution (u, v) has a linear growth at infinity, then it is one dimensional, that is, depending only on one variable. The main ingredient is an improvement of flatness estimate, which is achieved by the harmonic approximation technique adapted in the singularly perturbed situation.

Keywords: elliptic systems, phase separation, one dimensional symmetry, harmonic approximation.

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1 Introduction

In this paper, we continue our study in [18] on the De Giorgi type conjecture, i.e. one dimensional symmetry problem for solutions of the following two component elliptic system in \mathbb{R}^n :

$$\Delta u = uv^2, \quad \Delta v = vu^2, \quad u, v > 0 \quad \text{in } \mathbb{R}^n. \quad (1.1)$$

We remove the energy minimizing condition in [18] and prove the one dimensional symmetry only under the linear growth condition. More precisely we prove

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Theorem 1.1. *If (u, v) is a solution of the problem (1.1), and there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$,*

$$u(x) + v(x) \leq C(1 + |x|), \quad (1.2)$$

then after a suitable rotation in \mathbb{R}^n ,

$$u(x) \equiv u(x_n), \quad v(x) \equiv v(x_n).$$

The linear growth condition is sharp, as shown by the examples constructed in [4], where $u - v$ is asymptotic to a homogeneous harmonic polynomial of degree $d \geq 2$. For more discussions on (1.1), we refer to [3, 4, 6, 14].

Through suitable rescalings, the problem (1.1) is closely related to the following singularly perturbed problem (see Theorem 2.1 below)

$$\begin{cases} \Delta u_\kappa = \kappa u_\kappa v_\kappa^2, \\ \Delta v_\kappa = \kappa v_\kappa u_\kappa^2, \end{cases} \quad (1.3)$$

which is used to describe the “phase separation” phenomena. When $\kappa \rightarrow +\infty$, the convergence of solutions (u_κ, v_κ) of (1.3) and their singular limit were studied by Caffarelli and Lin [6], Noris-Tavares-Terracini-Verzini [14] and Tavares-Terracini [16] (see also Dancer-Zhang and the author [9]).

The main ingredient of our proof is an improvement of flatness estimate for the singular perturbation problem (1.3), which is achieved by the blow up (harmonic approximation) technique. This type of arguments, first introduced by De Giorgi in his work on the regularity of minimal hypersurfaces [10], are by now classical in the elliptic regularity theory. It plays an important role in the establishment of many ε -regularity theorems, such as in the theory of stationary varifolds (cf. Allard [1], see also [13, Section 6.5] for an account), harmonic maps (cf. L. Simon [17]) and nonlinear elliptic systems (the indirect method, see for example Chen-Wu [7, Chapter 12]), just to name a few examples.

In singular perturbation problems, Savin’s proof of the De Giorgi conjecture for Allen-Cahn equation [15] also uses an improvement of flatness estimate and the harmonic approximation type argument. However, there the quantity to be improved is different from the classical *energy* quantity. Indeed, the method developed in [15] is mainly on the viscosity (or Krylov-Safonov) side and corresponds to the Harnack inequality approach to the regularity of minimal hypersurfaces as developed in Caffarelli-Cordoba [5].

In this paper we will explore some aspects of harmonic approximation arguments in the singular perturbation problem (1.3), from the *variational* side. Thus in our estimate we still use an energy type quantity, which is similar to the *excess* used in Allard’s regularity theory. In this sense, our method may be viewed as a direct generalization of the classical harmonic approximation technique in this singular perturbation problem.

However, in order to get a harmonic function in the blow up limit, we use the stationary condition arising from the equation, but not the equation (1.3) itself. Let us first recall the stationary condition. Given a $\kappa > 0$ fixed, any solution of (1.3), (u_κ, v_κ) is smooth. Let Y be a smooth vector field with compact support, then by considering domain variations in the form

$$u_\kappa^t(x) := u_\kappa(x + tY(x)), \quad v_\kappa^t(x) := v_\kappa(x + tY(x)), \text{ for } |t| \text{ small,}$$

we have

$$\frac{d}{dt} \int (|\nabla u_\kappa^t(x)|^2 + |\nabla v_\kappa^t(x)|^2 + \kappa u_\kappa^t(x)^2 v_\kappa^t(x)^2) dx \Big|_{t=0} = 0.$$

Through some integration by parts we obtain the stationary condition for (u_κ, v_κ) ,

$$\int (|\nabla u_\kappa|^2 + |\nabla v_\kappa|^2 + \kappa u_\kappa^2 v_\kappa^2) \operatorname{div} Y - 2DY(\nabla u_\kappa, \nabla u_\kappa) - 2DY(\nabla v_\kappa, \nabla v_\kappa) = 0. \quad (1.4)$$

Here div is the divergence operator, and for a function u ,

$$DY(\nabla u, \nabla u) = \sum_{\alpha, \beta=1}^n \frac{\partial Y^\alpha}{\partial x_\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial u}{\partial x_\beta}.$$

For the problem (1.3), we have better control and convergence on the energy level, while the equation itself is badly behaved. This is why we choose to blow up the stationary condition to get a harmonic function in the limit. Here we would like to mention that the stationary condition appears more naturally in some other singular perturbation problems, such as the Allen-Cahn model (cf. Hutchinson-Tonegawa [12]) and the Ginzburg-Landau model (cf. Bethuel-Brezis-Orlandi [2]). In these problems, the stationary condition is directly linked to the limit problem, i.e. the stationary condition for varifolds (in the sense of Allard [1]).

In the remaining part of this paper, a solution (u, v) of the problem (1.1) will be fixed. We use the notation $a_\kappa = O(b_\kappa)$, if there exists a constant C such that, as $\kappa \rightarrow +\infty$,

$$|a_\kappa| \leq C b_\kappa,$$

and we say $a_\kappa = o(b_\kappa)$ if

$$\lim_{\kappa \rightarrow +\infty} \frac{a_\kappa}{b_\kappa} = 0.$$

We use C to denote various universal constants, which are independent of the base point $x \in \mathbb{R}^n$ and the radius R . (In some cases it depends on the solution itself.) It may be different from line to line. H^s is used to denote the s -dimensional Hausdorff measure.

2 The improvement of flatness

First, to explain why our main Theorem 1.1 is related to a singular perturbation problem, let us recall the following result, which is essentially [18, Lemma 5.2].

Theorem 2.1. *For any $\varepsilon > 0$, there exists an R_0 such that if $R \geq R_0$ and $x_0 \in \{u = v\}$, by defining*

$$u_\kappa(x) := \frac{1}{R}u(x_0 + Rx), \quad v_\kappa(x) := \frac{1}{R}v(x_0 + Rx), \quad (2.1)$$

there exists a constant c_0 independent of $x_0 \in \{u = v\}$ and R , and a vector e satisfying $|e| \geq c_0$, such that

$$\int_{B_1(0)} |\nabla u_\kappa - \nabla v_\kappa - e|^2 \leq \varepsilon^2.$$

Note that (u_κ, v_κ) satisfies (1.3) with $\kappa = R^4$. For a proof of this theorem see [18, Lemma 5.2]. The only new point is that, if

$$\sup_{B_2(0)} |u_\kappa - v_\kappa - e \cdot x|^2 \leq \varepsilon^2,$$

then $u_\kappa - v_\kappa$ is also close to $e \cdot x$ in $H^1(B_1(0))$ topology. This can be proved by a contradiction argument, using the H^1 strong convergence for solutions of (1.3) (cf. [14, Theorem 1.2]). Note that we can replace the global uniform Hölder estimate used in [14] by the interior uniform Hölder estimate [18, Theorem 2.6].

Throughout this paper, (u_κ, v_κ) always denotes a solution defined as in (2.1). The following improvement of decay estimate will be the main ingredient in our proof of Theorem 1.1.

Theorem 2.2. *There exist four universal constants $\theta \in (0, 1/2)$, ε_0 small and $K_0, C(n)$ large such that, if (u_κ, v_κ) is a solution of (1.3) in $B_1(0)$, satisfying*

$$\int_{B_1(0)} |\nabla u_\kappa - \nabla v_\kappa - e|^2 = \varepsilon^2 \leq \varepsilon_0^2, \quad (2.2)$$

where e is a vector satisfying $|e| \geq c_0/2$, and $\kappa^{1/4}\varepsilon^2 \geq K_0$, then there exists another vector \tilde{e} , with

$$|\tilde{e} - e| \leq C(n)\varepsilon,$$

such that

$$\theta^{-n} \int_{B_\theta(0)} |\nabla u_\kappa - \nabla v_\kappa - \tilde{e}|^2 \leq \frac{1}{2}\varepsilon^2.$$

The proof will be given later. Note that this theorem is not a local result. It depends on the global Lipschitz estimate established in [18], which is stated for solutions of (1.1) defined on the entire space \mathbb{R}^n .

This decay estimate can be used to prove

Theorem 2.3. *There exists a constant $C > 0$ such that, for any $x \in \{u = v\}$ and $R > 1$, there exists a vector $e_{x,R}$, with*

$$|e_{x,R}| \geq c_0/2,$$

such that

$$\int_{B_R(x)} |\nabla u - \nabla v - e_{x,R}|^2 \leq CR^{n-1}.$$

Proof. Fix an $R > 1$ and $x_0 \in \{u = v\}$, which we assume to be the origin 0. For each $i > 0$, denote

$$R_i := R\theta^{-i}.$$

Let E_i and the vector e_i be defined by

$$E_i := \min_{e \in \mathbb{R}^n} R_i^{1-n} \int_{B_{R_i}(0)} |\nabla u - \nabla v - e|^2 = R_i^{1-n} \int_{B_{R_i}(0)} |\nabla u - \nabla v - e_i|^2.$$

Note that for any fixed e ,

$$\begin{aligned} R_i^{1-n} \int_{B_{R_i}(0)} |\nabla u - \nabla v - e|^2 &\leq R_i^{1-n} \int_{B_{\theta^{-1}R_i}(0)} |\nabla u - \nabla v - e|^2 \\ &\leq \theta^{1-n} (\theta^{-1}R_i)^{1-n} \int_{B_{\theta^{-1}R_i}(0)} |\nabla u - \nabla v - e|^2. \end{aligned}$$

Hence we always have

$$E_i \leq \theta^{1-n} E_{i+1}. \quad (2.3)$$

Furthermore, since (see [18, Theorem 5.1])

$$\sup_{\mathbb{R}^n} (|\nabla u| + |\nabla v|) < +\infty,$$

there exists a constant C , which is independent of i , such that

$$E_i \leq C\theta^{-i}. \quad (2.4)$$

By Theorem 2.1, for any sequence $i \rightarrow +\infty$, there exists a subsequence (still denoted by i) such that

$$u_i(x) := R_i^{-1}u(R_i x) \rightarrow (e \cdot x)^+, \quad v_i(x) := R_i^{-1}v(R_i x) \rightarrow (e \cdot x)^-.$$

Here e is a vector in \mathbb{R}^n satisfying $|e| \geq c_0$, and the convergence is in $C_{loc}(\mathbb{R}^n)$ and also in $H_{loc}^1(\mathbb{R}^n)$. Note that (u_i, v_i) satisfies (1.3) with $\kappa_i = R_i^4$.

Indeed, by Theorem 2.1, if $R_i \geq R_0$, where R_0 is a constant depending only on ε_0 , there exists a vector \bar{e}_i with $|\bar{e}_i| \geq c_0$ such that

$$\int_{B_{R_i}(0)} |\nabla u - \nabla v - \bar{e}_i|^2 \leq \varepsilon_0^2 R_i^n.$$

By definition, if we replace \bar{e}_i by e_i , we can get the same estimate. Thus by Theorem 2.2, if we also have $E_{i+1} \geq K_0$, or equivalently,

$$\int_{B_1(0)} |\nabla u_{i+1} - \nabla v_{i+1} - e_{i+1}|^2 \geq K_0 R_{i+1}^{-1} = K_0 \kappa_{i+1}^{-\frac{1}{4}},$$

then there exists another vector \tilde{e}_{i+1} so that

$$\theta^{-n} \int_{B_\theta(0)} |\nabla u_{i+1} - \nabla v_{i+1} - \tilde{e}_{i+1}|^2 \leq \frac{1}{2} \int_{B_1(0)} |\nabla u_{i+1} - \nabla v_{i+1} - e_{i+1}|^2.$$

This can be rewritten as

$$\begin{aligned} E_i &\leq R_i^{1-n} \int_{B_{R_i}(0)} |\nabla u - \nabla v - \tilde{e}_{i+1}|^2 \\ &= \theta^{-n} R_i \int_{B_\theta(0)} |\nabla u_{i+1} - \nabla v_{i+1} - \tilde{e}_{i+1}|^2 \\ &\leq \frac{R_i}{2} \int_{B_1(0)} |\nabla u_{i+1} - \nabla v_{i+1} - e_{i+1}|^2 \\ &= \frac{1}{2} R_i R_{i+1}^{-n} \int_{B_{R_{i+1}}(0)} |\nabla u - \nabla v - e_{i+1}|^2 \\ &= \frac{\theta}{2} E_{i+1}. \end{aligned} \tag{2.5}$$

Now we claim that for all $i \geq \min\{\frac{\log R_0 - \log R}{|\log \theta_0|}, 1\}$,

$$E_i \leq \theta^{1-n} K_0. \tag{2.6}$$

Assume by the contrary, there exists an $i_0 \geq \min\{\frac{\log R_0 - \log R}{|\log \theta_0|}, 1\}$ such that $E_{i_0} > \theta^{1-n} K_0$. First by (2.3),

$$E_{i_0+1} \geq \theta^{n-1} E_{i_0} > K_0.$$

Thus the assumptions of Theorem 2.2 are satisfied and we have (2.5), which says

$$E_{i_0+1} \geq \frac{2}{\theta} E_{i_0} \geq E_{i_0} > \theta^{1-n} K_0.$$

This can be iterated, and we get, for any $j \geq 0$,

$$E_{i_0+j+1} \geq \frac{2}{\theta} E_{i_0+j} \geq E_{i_0} \left(\frac{2}{\theta} \right)^{j+1}.$$

However, since $2/\theta > 1/\theta$, this contradicts (2.4) if j is large enough. Note that the constant $\theta^{1-n} K_0$ in (2.6) is independent of the base point $x_0 \in \{u = v\}$ and the radius R . Thus we get (2.6) for any $R \geq R_0$ and $x \in \{u = v\}$. Then by choosing a larger constant, this can be extended to cover $[1, R_0]$ if we note the global Lipschitz bound of u and v .

We have shown the existence of $e_{x,R}$ for any $x \in \{u = v\}$ and $R > 1$. The lower bound for $|e_{x,R}|$ can be proved as in the proof of Theorem 2.1 by using the Alt-Caffarelli-Friedman inequality (see [18, Theorem 4.3]). \square

With this theorem in hand, we can use $e_{x,R} \cdot (y - x)$ to replace the harmonic replacement $\varphi_{R,x}$ in [18, Section 7]. The following arguments to prove Theorem 1.1 are exactly the same one in [18, Section 8 and 9].

The remaining part of this paper will be devoted to the proof of Theorem 2.2.

3 Some a priori estimates

In this section, we present some a priori estimates for the solution (u, v) . These estimates show that various quantities, when integrated on $B_R(x)$, have a growth bound as R^{n-1} . This is exactly what we expect for one dimensional solutions. Several estimates from [18] will be needed in this section.

Lemma 3.1. *There exist two positive constants C and M , such that for any $R > CM$ and $t \geq M$,*

$$H^{n-1}(B_R \cap \{u = t\}) \leq CR^{n-1}.$$

Proof. First, by [18, Lemma 5.2 and Lemma 5.4], there exists a constant $c(M) > 0$ such that,

$$|\nabla u| \geq c(M) \quad \text{on } \{u = t\}. \quad (3.1)$$

Since u is smooth, by the implicit function theorem $\{u = t\}$ is a smooth hypersurface.

Now

$$H^{n-1}(B_R \cap \{u = t\}) \leq c(M)^{-1} \int_{B_R \cap \{u=t\}} |\nabla u|. \quad (3.2)$$

Note that on $\{u = t\}$

$$|\nabla u| = -\frac{\partial u}{\partial \nu},$$

where ν is the unit normal vector of $\{u = t\}$ pointing to $\{u < t\}$. Then by the divergence theorem

$$\begin{aligned} -\int_{B_R \cap \{u=t\}} \frac{\partial u}{\partial \nu} &= \int_{\partial B_R \cap \{u>t\}} \frac{\partial u}{\partial r} - \int_{B_R \cap \{u>t\}} \Delta u \\ &\leq \int_{\partial B_R \cap \{u>t\}} |\nabla u| + \int_{B_R} \Delta u \\ &\leq 2 \int_{\partial B_R} |\nabla u| \leq CR^{n-1}. \end{aligned}$$

Here we have used the global Lipschitz continuity of u , cf. [18, Theorem 5.1]. \square

The same results also hold for v and $u - v$, which we do not repeat here. Next we give a measure estimate for the transition part $\{u \leq T, v \leq T\}$.

Lemma 3.2. *For any $T > 1$, there exists a constant $C(T) > 0$, such that for any $R > 1$ and $x \in \mathbb{R}^n$,*

$$H^n(B_R(x) \cap \{u \leq T, v \leq T\}) \leq C(T)R^{n-1}.$$

Proof. First we have the

Claim. For each $T > 1$, there exists a $c(T) > 0$ such that, if $x_0 \in \{u \leq T, v \leq T\}$, then

$$u(x_0) \geq c(T), \quad v(x_0) \geq c(T). \quad (3.3)$$

By assuming this claim, we get

$$H^n(B_R(x) \cap \{u \leq T, v \leq T\}) \leq c(T)^{-4} \int_{B_R(x) \cap \{u \leq T, v \leq T\}} u^2 v^2 \leq C(T)R^{n-1},$$

where in the last inequality we have used [18, Lemma 6.4].

To prove the claim, first we note that, there exists a constant $C_1(T)$ such that

$$\text{dist}(x_0, \{u = v\}) \leq C_1(T). \quad (3.4)$$

Indeed, if $\text{dist}(x_0, \{u = v\}) \geq L$ (L large to be chosen), take $y_0 \in \{u = v\}$ to realize this distance and define

$$\tilde{u}(x) = \frac{1}{L}u(y_0 + Lx), \quad \tilde{v}(x) = \frac{1}{L}v(y_0 + Lx).$$

Then by [18, Lemma 5.2], there exists a vector e and a universal constant C , with

$$\frac{1}{C} \leq |e| \leq C,$$

such that

$$|\tilde{u}(x) - (e \cdot x)^+| + |\tilde{v}(x) - (e \cdot x)^-| \leq h(L),$$

where $h(L)$ is small if L large enough.

Without loss of generality we can assume $B_1(L^{-1}(x_0 - y_0)) \subset \{\tilde{u} > \tilde{v}\}$. By a geometric consideration, we have

$$L^{-1}(x_0 - y_0) \cdot e \geq \frac{1}{2C}.$$

Consequently,

$$\tilde{u}(L^{-1}(x_0 - y_0)) \geq L^{-1}(x_0 - y_0) \cdot e - h(L) \geq \frac{1}{4C}.$$

Thus $u(x_0) > T$ if L large, which is a contradiction.

After establishing (3.4), we can use the standard Harnack inequality and [18, Lemma 4.7] to deduce the claimed (3.3). \square

Lemma 3.3. *There exists a constant $C > 0$, such that for any $R > 1$ and $x \in \mathbb{R}^n$,*

$$\int_{B_R(x)} |\nabla u| |\nabla v| \leq CR^{n-1}.$$

Proof. Fix a $T > 0$, which will be determined below. (It is independent of x and R .) We divide the estimate into three parts, $\{u \leq T, v \leq T\}$, $\{u > T\}$ and $\{v > T\}$. Note that if T is large enough, by [18, Lemma 6.1], these three parts are disjoint.

First in $B_R(x) \cap \{u \leq T, v \leq T\}$, by the global Lipschitz continuity of u and v [18, Theorem 5.1] and the previous lemma, we have

$$\int_{B_R(x) \cap \{u \leq T, v \leq T\}} |\nabla u| |\nabla v| \leq CH^n(B_R(x) \cap \{u \leq T, v \leq T\}) \leq CR^{n-1}. \quad (3.5)$$

If T large, in $\{u > T\}$, $|\nabla u| \geq c(T) > 0$ for a constant $c(T)$ depending only on T (cf. the proof of Lemma 3.1). Furthermore, by the proof of [18, Lemma 6.3], there exists a constant C such that

$$|\nabla v| \leq Ce^{-\frac{u}{C}} \quad \text{in } \{u > T\}.$$

Then by the co-area formula and Lemma 3.1,

$$\int_{B_R(x) \cap \{u > T\}} |\nabla u| |\nabla v| = \int_T^{+\infty} \left(\int_{B_R \cap \{u=t\}} |\nabla v| \right) dt$$

$$\begin{aligned}
&\leq C \int_T^{+\infty} e^{-ct} H^{n-1}(B_R \cap \{u = t\}) dt \\
&\leq CR^{n-1}.
\end{aligned}$$

The same estimate holds for $\{v > T\}$. Putting these together we can finish the proof. \square

Lemma 3.4. *There exists a constant $C > 0$, such that for any $R > 1$ and $x \in \mathbb{R}^n$,*

$$\int_{B_R(x)} uv^3 + vu^3 \leq CR^{n-1}.$$

Proof. We still choose a $T > 0$, which will be determined below, and divide the estimate into three parts, $\{u \leq T, v \leq T\}$, $\{u > T\}$ and $\{v > T\}$.

By the proof of [18, Lemma 6.1], we still have

$$uv^3 + vu^3 \leq C \quad \text{in } \mathbb{R}^n.$$

Then in $B_R(x) \cap \{u \leq T, v \leq T\}$,

$$\int_{B_R(x) \cap \{u \leq T, v \leq T\}} uv^3 + vu^3 \leq CH^n(B_R(x) \cap \{u \leq T, v \leq T\}) \leq CR^{n-1}.$$

Next, by the proof of [18, Lemma 6.1], there exists a constant C such that

$$uv^3 + vu^3 \leq Ce^{-\frac{u}{C}} \quad \text{in } \{u > T\}.$$

Then by the co-area formula and the lower bound of $|\nabla u|$ in $\{u > T\}$ (i.e. (3.1)),

$$\begin{aligned}
\int_{B_R(x) \cap \{u > T\}} uv^3 + vu^3 &= \int_M^{+\infty} \left(\int_{B_R \cap \{u=t\}} \frac{uv^3 + vu^3}{|\nabla u|} \right) dt \\
&\leq C \int_M^{+\infty} e^{-ct} H^{n-1}(B_R \cap \{u = t\}) dt \\
&\leq CR^{n-1}.
\end{aligned}$$

The same estimate holds for $\{v > T\}$. Putting these together we can finish the proof. \square

4 Blow up the stationary condition

In this section and the next one, we prove Theorem 2.2. We argue by contradiction, so assume that as $\kappa \rightarrow +\infty$, there exists a sequence of solutions (u_κ, v_κ) satisfying the conditions but not the conclusions in that theorem, that is,

$$\int_{B_1(0)} |\nabla u_\kappa - \nabla v_\kappa - e|^2 = \varepsilon_\kappa^2 \rightarrow 0, \quad (4.1)$$

where e is a vector satisfying $|e| \geq c_0$ (Without loss of generality, we can assume that $e = (0, \dots, 0, |e|) = |e|e_n$), but for any vector \tilde{e} satisfying (here the constant $C(n)$ will be determined later)

$$|\tilde{e} - e| \leq C(n)\varepsilon_\kappa, \quad (4.2)$$

we must have (θ will be determined later)

$$\theta^{-n} \int_{B_\theta(0)} |\nabla u_\kappa - \nabla v_\kappa - \tilde{e}|^2 \geq \frac{1}{2}\varepsilon_\kappa^2. \quad (4.3)$$

Moreover, we also assume that,

$$\lim_{\kappa \rightarrow +\infty} \kappa^{1/4} \varepsilon_\kappa^2 = +\infty. \quad (4.4)$$

We will derive a contradiction from these assumptions.

The first step, which will be done in this section, is to show that the blow up sequence

$$\frac{u_\kappa - v_\kappa - e \cdot x}{\varepsilon_\kappa},$$

converges to a harmonic function in some weak sense.

Recall that (u_κ, v_κ) satisfies the stationary condition

$$\int_{B_1(0)} (|\nabla u_\kappa|^2 + |\nabla v_\kappa|^2 + \kappa u_\kappa^2 v_\kappa^2) \operatorname{div} Y - 2DY(\nabla u_\kappa, \nabla u_\kappa) - 2DY(\nabla v_\kappa, \nabla v_\kappa) = 0. \quad (4.5)$$

Since there exists an $R > 0$ such that $\kappa = R^4$, and

$$u_\kappa(x) = R^{-1}u(Rx), \quad v_\kappa(x) = R^{-1}v(Rx),$$

by [18, Lemma 6.4],

$$\int_{B_1(0)} \kappa u_\kappa^2 v_\kappa^2 = R^{-n} \int_{B_R(0)} u^2 v^2 \leq CR^{-1} = C\kappa^{-\frac{1}{4}}.$$

Similarly, by Lemma 3.3,

$$\int_{B_1(0)} |\nabla u_\kappa| |\nabla v_\kappa| = R^{-n} \int_{B_R(0)} |\nabla u| |\nabla v| \leq C\kappa^{-\frac{1}{4}}.$$

Then by a direct expansion, we get

$$\int_{B_1(0)} |\nabla(u_\kappa - v_\kappa)|^2 \operatorname{div} Y - 2DY(\nabla(u_\kappa - v_\kappa), \nabla(u_\kappa - v_\kappa)) = O(\kappa^{-\frac{1}{4}}). \quad (4.6)$$

Now let

$$w_\kappa := \frac{u_\kappa - v_\kappa - e \cdot x}{\varepsilon_\kappa} - \lambda_\kappa,$$

where λ_κ is chosen so that

$$\int_{B_1(0)} w_\kappa = 0. \quad (4.7)$$

Then

$$\int_{B_1(0)} |\nabla w_\kappa|^2 = 1,$$

and by noting (4.7) we can apply the Poincare inequality to get

$$\int_{B_1(0)} w_\kappa^2 \leq C(n).$$

Hence after passing to a subsequence of κ , we can assume that w_κ converges to w , weakly in $H^1(B_1(0))$ and strongly in $L^2(B_1(0))$.

Substituting w_κ into (4.6), we obtain

$$\begin{aligned} O(\kappa^{-\frac{1}{4}}) &= \varepsilon_\kappa^2 \int_{B_1(0)} [|\nabla w_\kappa|^2 \operatorname{div} Y - 2DY(\nabla w_\kappa, \nabla w_\kappa)] \\ &\quad + 2\varepsilon_\kappa \int_{B_1(0)} [\nabla w_\kappa \cdot e \operatorname{div} Y - DY(\nabla w_\kappa, e) - DY(e, \nabla w_\kappa)] \\ &\quad + \int_{B_1(0)} [|e|^2 \operatorname{div} Y + 2DY(e, e)]. \end{aligned} \quad (4.8)$$

The last integral equals 0. Integrating by parts, we also have

$$\int_{B_1(0)} [\nabla w_\kappa \cdot e \operatorname{div} Y - DY(\nabla w_\kappa, e) - DY(e, \nabla w_\kappa)] = |e| \int_{B_1(0)} Y^n \Delta w_\kappa.$$

Substituting this into (4.8), we obtain

$$2|e| \int_{B_1(0)} Y^n \Delta w_\kappa = -\varepsilon_\kappa \int_{B_1(0)} [|\nabla w_\kappa|^2 \operatorname{div} Y - 2DY(\nabla w_\kappa, \nabla w_\kappa)] + O(\kappa^{-\frac{1}{4}} \varepsilon_\kappa^{-1}). \quad (4.9)$$

The right hand sides goes to 0 as $\kappa \rightarrow 0$, thanks to our assumptions that $\varepsilon_\kappa \rightarrow 0$ and $\kappa^{1/4} \varepsilon_\kappa \rightarrow +\infty$. After passing to the limit in the above equality, we see

$$\int_{B_1(0)} Y^n \Delta w = 0.$$

Since Y^n can be any function in $C_0^\infty(B_1(0))$, by standard elliptic theory we get

Proposition 4.1. *w is a harmonic function.*

5 Strong convergence of the blow up sequence

In this section, we prove the strong convergence of w_κ in $H_{loc}^1(B_1(0))$. With some standard estimates on harmonic functions, this will give the decay estimate Theorem 2.2.

In order to prove the strong convergence of w_κ in $H_{loc}^1(B_1(0))$, we define the defect measure μ by

$$|\nabla w_\kappa|^2 dx \rightharpoonup |\nabla w|^2 dx + \mu \quad \text{weakly as measures.}$$

By the weak convergence of w_κ in $H^1(B_1(0))$, μ is a positive Radon measure. Furthermore, w_κ converges strongly in $H_{loc}^1(B_1(0))$ if and only if $\mu = 0$ in $B_1(0)$.

First we note the fact that

Lemma 5.1. *The support of μ lies in the hyperplane $\{x_n = 0\}$.*

Proof. By [18, Theorem 2.7] and (4.1), as $\kappa \rightarrow +\infty$, (u_κ, v_κ) converges to $|e|(x_n^+, x_n^-)$ uniformly in $B_1(0)$. For any $h > 0$, if κ large,

$$u_\kappa \geq h, \quad v_\kappa \leq h \quad \text{in } \{x_n > 2|e|^{-1}h\}.$$

Then

$$\Delta v_\kappa \geq \kappa h^2 v_\kappa \quad \text{in } \{x_n > 2|e|^{-1}h\},$$

and by [8, Lemma 4.4],

$$v_\kappa \leq C(n)h e^{-\frac{\kappa^{1/2}h^2}{C(n)}} \quad \text{in } \{x_n > 3|e|^{-1}h\}.$$

Note that u_κ and v_κ are uniformly bounded in $B_1(0)$ because they are nonnegative, subharmonic. Then by definition

$$\Delta w_\kappa = \varepsilon_\kappa^{-1} [\kappa v_\kappa^2 u_\kappa - \kappa u_\kappa^2 v_\kappa] = O(\varepsilon_\kappa^{-1} e^{-\frac{\kappa^{1/2}h^2}{C(n)}}) \rightarrow 0,$$

uniformly in $\{x_n \geq 4|e|^{-1}h\}$, thanks to our assumption that $\varepsilon_\kappa \gg \kappa^{-1/8}$. Applying standard interior $W^{2,2}$ estimates, together with the assumption (4.1), and noting that w_κ is uniformly bounded in $H_{loc}^1(B_1(0))$, we see

$$\int_{B_{4/5}(0) \cap \{x_n \geq 5|e|^{-1}h\}} |D^2 w_\kappa|^2$$

is uniformly bounded. By Rellich compactness theorem, ∇w_κ converges to ∇w strongly in $L_{loc}^2(B_{4/5} \cap \{x_n \geq 5h\})$. In other words, the support of μ lies in $\{x_n \leq 5|e|^{-1}h\}$. We can get the other side estimate and also let $h \rightarrow 0$ to finish the proof. \square

For any $\eta \in C_0^\infty(B_1(0))$, by Lemma 3.4 and [18, Lemma 6.4],

$$\begin{aligned}
& \int_{B_1(0)} \Delta(u_\kappa - v_\kappa)(u_\kappa - v_\kappa) \eta^2 \\
&= \int_{B_1(0)} (2\kappa u_\kappa^2 v_\kappa^2 - \kappa u_\kappa v_\kappa^3 - \kappa v_\kappa u_\kappa^3) \eta^2 \\
&= R^{-n} \int_{B_R(0)} [2u(y)^2 v(y)^2 - u(y)v(y)^3 - v(y)u(y)^3] \eta(R^{-1}y)^2 dy \\
&= O(R^{-1}) = O(\kappa^{-1/4}).
\end{aligned} \tag{5.1}$$

Substituting

$$\Delta(u_\kappa - v_\kappa) = \varepsilon_\kappa \Delta w_\kappa, \quad u_\kappa - v_\kappa = |e|x_n + \varepsilon_\kappa w_\kappa + \varepsilon_\kappa \lambda_\kappa,$$

into this, we get

$$\int_{B_1(0)} \varepsilon_\kappa^2 \Delta w_\kappa w_\kappa \eta^2 + \varepsilon_\kappa |e| \Delta w_\kappa \eta^2 x_n + \varepsilon_\kappa^2 \lambda_\kappa \Delta w_\kappa \eta^2 = O(\kappa^{-1/4}). \tag{5.2}$$

On the other hand, by taking $Y = (0, \dots, 0, \eta^2 x_n)$ in (4.9), we have ¹

$$\begin{aligned}
& 2|e| \int_{B_1(0)} \Delta w_\kappa \eta^2 x_n \\
&= -\varepsilon_\kappa \int_{B_1(0)} |\nabla w_\kappa|^2 \left(\eta^2 + 2\eta \frac{\partial \eta}{\partial x_n} x_n \right) - 2 \sum_{i=1}^n \frac{\partial w_\kappa}{\partial x_n} \frac{\partial w_\kappa}{\partial x_i} \frac{\partial}{\partial x_i} (\eta^2 x_n) + O(\kappa^{-\frac{1}{4}} \varepsilon_\kappa^{-1}).
\end{aligned} \tag{5.3}$$

Substituting this into (5.2) we see

$$\begin{aligned}
& \int_{B_1(0)} \Delta w_\kappa w_\kappa \eta^2 - \frac{1}{2} |\nabla w_\kappa|^2 \left(\eta^2 + 2\eta \frac{\partial \eta}{\partial x_n} x_n \right) \\
&+ \sum_{i=1}^n \frac{\partial w_\kappa}{\partial x_n} \frac{\partial w_\kappa}{\partial x_i} \frac{\partial}{\partial x_i} (\eta^2 x_n) + \lambda_\kappa \Delta w_\kappa \eta^2 = O(\kappa^{-\frac{1}{4}} \varepsilon_\kappa^{-2}).
\end{aligned} \tag{5.4}$$

Concerning the last term in this integral we claim that

¹This inequality could be understood as a Caccioppoli type inequality, which is similar to the one in Allard's regularity theory for stationary varifolds, see [1, Lemma 8.11] and [13, Lemma 6.5.5]. The choice of Y has a more direct geometric meaning (as a vector field in normal directions) in that setting. However, we note that (5.1) can also be understood as a Caccioppoli type inequality, as in (for example) De Giorgi-Nash-Moser theory for linear elliptic equations in divergence form.

Lemma 5.2. *For any $\eta \in C_0^\infty(B_1(0))$,*

$$\lim_{\kappa \rightarrow +\infty} \int_{B_1(0)} \lambda_\kappa \Delta w_\kappa \eta^2 = 0.$$

Proof. By the definition of λ_κ (see (4.7)),

$$\begin{aligned} \lambda_\kappa &= \frac{1}{\varepsilon_\kappa H^n(B_1(0))} \int_{B_1(0)} (u_\kappa - v_\kappa - e \cdot x) \\ &= \frac{1}{\varepsilon_\kappa H^n(B_1(0))} \int_{B_1(0)} (u_\kappa - v_\kappa). \end{aligned}$$

By Theorem 2.1, $u_\kappa - v_\kappa$ converges uniformly on $B_1(0)$ to the harmonic function $e \cdot x$, thanks to the assumption that we always have (see the definition of u_κ and v_κ , (2.1))

$$u_\kappa(0) - v_\kappa(0) = 0.$$

This then implies that

$$\lim_{\kappa \rightarrow +\infty} \int_{B_1(0)} (u_\kappa - v_\kappa) = 0.$$

Hence we have

$$\lambda_\kappa = o(\varepsilon_\kappa^{-1}). \quad (5.5)$$

On the other hand, by choosing the vector field $Y = (0, \dots, 0, \eta^2)$ in (4.9), we have

$$\int_{B_1(0)} \eta^2 \Delta w_\kappa = O(\varepsilon_\kappa), \quad (5.6)$$

where we have used the uniform bound on $\|w_\kappa\|_{H^1(B_1(0))}$ and our assumption (4.4).

Combining (5.5) and (5.6) we get the required convergence. \square

Note that (4.4) says $\kappa^{-\frac{1}{4}} \varepsilon_\kappa^{-2} \rightarrow 0$. An integration by parts in the first term in (5.4) gives

$$\int_{B_1(0)} -2\eta w_\kappa \nabla w_\kappa \nabla \eta - \frac{3}{2} |\nabla w_\kappa|^2 \eta^2 - |\nabla w_\kappa|^2 \eta \frac{\partial \eta}{\partial x_n} x_n + \sum_{i=1}^n \frac{\partial w_\kappa}{\partial x_n} \frac{\partial w_\kappa}{\partial x_i} \frac{\partial}{\partial x_i} (\eta^2 x_n) \rightarrow 0. \quad (5.7)$$

Next we analyze the convergence in (5.7) term by term. By the weak convergence of ∇w_κ and strong convergence of w_κ in $L^2(B_1(0))$,

$$\int_{B_1(0)} \eta w_\kappa \nabla w_\kappa \nabla \eta \rightarrow \int_{B_1(0)} \eta w \nabla w \nabla \eta.$$

By the weak convergence of $|\nabla w_\kappa|^2 dx$,

$$\int_{B_1(0)} |\nabla w_\kappa|^2 \eta^2 \rightarrow \int_{B_1(0)} |\nabla w|^2 \eta^2 + \int_{B_1(0)} \eta^2 d\mu,$$

and

$$\begin{aligned} \int_{B_1(0)} |\nabla w_\kappa|^2 \eta \frac{\partial \eta}{\partial x_n} x_n &\rightarrow \int_{B_1(0)} |\nabla w|^2 \eta \frac{\partial \eta}{\partial x_n} x_n + \int_{B_1(0)} \eta \frac{\partial \eta}{\partial x_n} x_n d\mu \\ &= \int_{B_1(0)} |\nabla w|^2 \eta \frac{\partial \eta}{\partial x_n} x_n. \end{aligned}$$

Here we have used Lemma 5.1 and the fact that $\eta \frac{\partial \eta}{\partial x_n} x_n = 0$ on $\{x_n = 0\}$. Similarly,

$$\int_{B_1(0)} \sum_{i=1}^n \frac{\partial w_\kappa}{\partial x_n} \frac{\partial w_\kappa}{\partial x_i} \frac{\partial}{\partial x_i} (\eta^2 x_n) \rightarrow \int_{B_1(0)} \sum_{i=1}^n \frac{\partial w}{\partial x_n} \frac{\partial w}{\partial x_i} \frac{\partial}{\partial x_i} (\eta^2 x_n) + \int_{B_1(0)} \eta^2 d\mu^n,$$

where μ^n is the weak limit of the measures $(\frac{\partial w_\kappa}{\partial x_n})^2 dx - (\frac{\partial w}{\partial x_n})^2 dx$.

Substituting these into (5.7) we get

$$\begin{aligned} \int_{B_1(0)} -2\eta w \nabla w \nabla \eta - \frac{3}{2} |\nabla w|^2 \eta^2 - |\nabla w|^2 \eta \frac{\partial \eta}{\partial x_n} x_n + \sum_{i=1}^n \frac{\partial w}{\partial x_n} \frac{\partial w}{\partial x_i} \frac{\partial}{\partial x_i} (\eta^2 x_n) \\ - \frac{3}{2} \int_{B_1(0)} \eta^2 d\mu + \int_{B_1(0)} \eta^2 d\mu^n = 0. \end{aligned} \quad (5.8)$$

Since w is a harmonic function, an integration by parts gives

$$\int_{B_1(0)} -2\eta w \nabla w \nabla \eta = \int_{B_1(0)} |\nabla w|^2 \eta^2. \quad (5.9)$$

Note that w is smooth. Then by standard domain variation arguments, we also have a stationary condition for w , which says, for any smooth vector field Y with compact support in $B_1(0)$,

$$\int_{B_1(0)} |\nabla w|^2 \operatorname{div} Y - 2DY(\nabla w, \nabla w) = 0.$$

By taking $Y = (0, \dots, 0, \eta^2 x_n)$ in this equality, we obtain

$$\int_{B_1(0)} |\nabla w|^2 \left(\eta^2 + 2\eta \frac{\partial \eta}{\partial x_n} x_n \right) - 2 \sum_{i=1}^n \frac{\partial w}{\partial x_n} \frac{\partial w}{\partial x_i} \frac{\partial}{\partial x_i} (\eta^2 x_n) = 0.$$

Substituting this and (5.9) into (5.8), we get

$$-\frac{3}{2} \int_{B_1(0)} \eta^2 d\mu + \int_{B_1(0)} \eta^2 d\mu^n = 0. \quad (5.10)$$

By the weak convergence of w_κ in $H^1(B_1(0))$, we also have

$$|\nabla(w_\kappa - w)|^2 dx \rightharpoonup \mu,$$

$$\left(\frac{\partial w_\kappa}{\partial x_n} - \frac{\partial w}{\partial x_n} \right)^2 dx \rightharpoonup \mu^n.$$

From these we see for any $\eta \in C_0^\infty(B_1(0))$,

$$\int_{B_1(0)} \eta^2 d\mu \geq \int_{B_1(0)} \eta^2 d\mu^n.$$

Substituting this into (5.10), we see

$$\int_{B_1(0)} \eta^2 d\mu \leq 0.$$

Since μ is a positive Radon measure, this implies that $\mu = 0$, and hence the strong convergence of w_κ in $H_{loc}^1(B_1(0))$.

With these preliminary analysis we come to the

Proof of Theorem 2.2. Note that w is a harmonic function satisfying

$$\int_{B_1(0)} |\nabla w|^2 \leq 1.$$

By standard interior gradient estimates for harmonic functions, there exists a constant $C_1(n)$ depending only on the dimension n , such that for any $r \in (0, 1/2)$,

$$\int_{B_r(0)} |\nabla w - \nabla w(0)|^2 \leq C_1(n) r^{n+2}.$$

Here we have used the fact that each component of ∇w is harmonic. By the mean value property for harmonic functions, there exists another constant $C_2(n)$ still depending only on the dimension n , such that

$$|\nabla w(0)| \leq C_2(n).$$

With these choices, now we fix the constant $C(n)$ in (4.2) to be $2C_2(n)$.

Fix a $\theta \in (0, 1/2)$ so that

$$2C_1(n)\theta^2 \leq \frac{1}{4}.$$

Then by the strong convergence of w_κ to w in $H^1(B_{1/2})$, for κ large,

$$\theta^{-n} \int_{B_\theta(0)} |\nabla w_\kappa - \nabla w(0)|^2 \leq \theta^{-n} \int_{B_\theta(0)} |\nabla w - \nabla w(0)|^2 + \frac{1}{4} \leq \frac{1}{2}.$$

In other words,

$$\theta^{-n} \int_{B_\theta(0)} |\nabla(u_\kappa - v_\kappa) - (e + \varepsilon_\kappa \nabla w(0))|^2 \leq \frac{1}{2} \varepsilon_\kappa^2.$$

By our construction, $e + \varepsilon_\kappa \nabla w(0)$ satisfies (4.2). The above inequality contradicts (4.3) and we finish the proof of Theorem 2.2. \square

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